Instanton representation of Plebanski gravity: V. Riemannian structure and relation to metric GR

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Abstract

In this paper we solidify the relation of the instanton representation of Plebanski gravity to Einstein's metric theory, using the relation between nonabelian gauge theory and the intrinsic geometry of gauge invariant variables. It is found that this representation corresponds to a 3-dimensional space with nonmetricity and torsion. By imposing metricity by way of the Gauss' law constraint, we correlate this 3-dimensional space to a 4-dimensional spacetime geometry which implies the Einstein–Hilbert action. We have generalized the Einstein spaces derived by previous authors, within the purley Yang–Mills context, to incorporate gravitational degrees of freedom.

1 Introduction

The aim of this paper is to establish a direct relationship between the instanton representation of Plebanski gravity and metric general relativity. In the instanton representation the basic phase space variables are $\Omega_{Inst}=(\Psi_{ae},A^a_i)$, the CDJ matrix and the self-dual $SO(3,C)$ connection. The CDJ matrix is $SO(3,C)\otimes SO(3,C)$- valued and is the basic momentum space variable, which encodes the gravitational degrees of freedom upon solution to the initial value constraints. It is shown in Paper II that the instanton representation can be derived directly from Plebanski theory subject to metricity, and also from the nondegenerate sector of the Ashtekar variables. Paper II also shows how the Einstein equations of motion follow, modulo the initial value constraints, from the instanton representation. These equations imply that a solution should be constructible directly from the phase space Ω_{Inst}, with the metric being a derived quantity expressed in terms of the physical degrees of freedom.

In the present paper we will provide further arguments that support the existence of such solutions by direct construction, by harnessing the relation of nonabelian gauge theory to intrinsic spatial geometry which has been exposed by previous authors within the Yang–Mills context. It happens that the configuration space variable for the instanton representation is a gauge connection A^a_i, which makes the analogous approach possible when gravitational degrees of freedom are present.[1] We will show how, starting from the instanton representation, one can obtain the Einstein–Hilbert action through a series of transformations. The main idea is to use the Ashtekar variables as a starting point in determining the Riemannian structure of the theory. Some of the main ideas behind this approach are contained in [1] and [2], where the authors uncover a natural spatial geometry encoded within $SU(2)$ and $SU(3)$ Yang-Mills theory. It is shown how using locally gauge-invariant quantities, one obtains a geometrization of these gauge theories. In particular, [1] derives an Einstein space with torsion.

Our approach in the present paper is to extend these concepts to incorporate gravitational degrees of freedom, using the Ashtekar variables as an intermediate stage in the derivation of the instanton representation in terms of its intrinsic spatial geometry. The outline of this paper is as follows. In section 3 we introduce the prescription for expressing the gauge connection in terms of an affine connection with spatial indices only. The standard procedure is then to use equality of the gauge curvature with the corresponding Riemann curvature tensor as a means for bringing out the hidden features

[1] These degrees of freedom are contained within the CDJ matrix Ψ_{ae}, which exists in general relativity but not in ordinary Yang–Mills theory.

of this geometry. To make the link to spaces more general than the Einstein space of [1], we introduce the CDJ Ansatz. This latter step defines the instanton representation, and can be seen as the generalization of [1] and [2] to incorporate the gravitational degrees of freedom.[2] We show that the the CDJ matrix is merely a representation of the Einstein tensor of a three dimensional space with nonmetricity and torsion. This is also re-phrased as a nonmetricity condition on the affine connection, wherein one separates the nonmetricity terms in the Riemann curvature. It turns out that the extent of nonmetricity is directly proportional to the Gauss' law constraint in the instanton representation.

By choosing the gauge connection to be the Ashtekar self-dual connection and counting degrees of freedom, we show that the torsion terms of this spatial geometry directly correlate to the extrinsic curvature of 3-space Σ. This enables one to make the link from this intrinsic 3-geometry to the 3+1 decomposition of a spacetime metric geometry. This is where the direct link to the Einstein–Hilbert action is established, which requires that the requisite canonical structure be put in place. This link is motivated in section 4 and proved in section 5. The first 4 sections of this paper establish the equivalence between the instanton representation and metric GR via the concept of this hidden spatial geometry. Section 5 solidifies this link from the gauge theory perspective, where the Einstein–Hilbert action is derived from the Hamiltonian constraint. Appendices A and B contain a short exerpt on the relation between metric variables and self-dual two forms by way of the 3+1 decomposition of spacetime.

[2]The CDJ matrix can be seen as the antiself-dual part of the Weyl curvature tensor, which encodes the nonlocal effects of gravitational radiation on curvature and the algbraic classification of spacetime.

2 From metric gravity into the instanton representation

The Einstein–Hilbert action for metric general relativity is given by

$$I_{EH}=\int_M d^4x\sqrt{-g}^{(4)}R[g], \tag{1}$$

where $g_{\mu\nu}$ is the spacetime metric. Using the results of Appendix A, (1) can be written as [3]

$$I_{EH}=\int dt\int_\Sigma d^3xN\sqrt{h}\big(^{(3)}R[h]+K_{ij}K^{ij}-(\mathrm{tr}K)^2\big)+\int_{\partial M}d^3x\sqrt{h}(\mathrm{tr}K), \tag{2}$$

whose Legendre transformation of (2) is given by

$$I_{EH}=\int dt\int_\Sigma d^3x\pi^{ij}\dot{h}_{ij}-N^\mu H_\mu. \tag{3}$$

In (3) h_{ij} is the 3-metric and π^{ij} is its conjugate momentum

$$\pi^{ij}=\sqrt{h}\big(K^{ij}-h^{ij}\mathrm{tr}K\big), \tag{4}$$

where K_{ij} and $^{(3)}R[h]$ are respectively the extrinsic and the instrinsic curvature of a 3-dimensional spatial hypersurface $\Sigma\subset M$. The constraints on the full Einstein–Hilbert metric phase space $\Omega_{EH}=(h_{ij},\pi^{ij})$ are the diffeomorphism and the Hamiltonian constraints $H_\mu=(H,H_i)$, given by

$$\pi^{ij}\pi_{ij}-\frac{1}{2}(\mathrm{tr}\pi)^2-\sqrt{h}^{(3)}R[h]=0;\quad H_i=\pi^{ij}_{;j}=0. \tag{5}$$

The unconstrained phase space contains $Dim(\Omega_{EH})=12$ real degrees of freedom per point. The implementation of (5) would reduce this to $Dim(\Omega_{EH})=$ 4 D.O.F. per point. However, (5) is nonpolynomial in the basic variables, which makes it intractable to solve in its present form.

The complex Ashtekar formalism introduces a left handed $SU(2)_-$ gauge connection A^a_i, which results in a transformation of the initial value constraints into polynomial form. In Ashtekar variables four of the seven constraints are given, including a cosmological constant, by (See e.g.[4], [5], [6])

$$H=\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_bB^k_c+\frac{\Lambda}{3}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^i_a\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c;\quad H_i=\epsilon_{ijk}\widetilde{\sigma}^j_aB^k_a, \tag{6}$$

respectively the Hamiltonian and the diffemorphism constraints. The object $B^i_a = \frac{1}{2}\epsilon^{ijk}F^a_{ij}[A]$ is the magnetic field derived from the field strength of A^a_i, given by

$$F^a_{ij} = \partial_i A^a_j - \partial_j A^a_i + f^{abc}A^b_i A^c_j. \tag{7}$$

The self-dual Ashtekar variables complexify the phase space and introduce an additional triple of constraints

$$G_a = D_i\widetilde{\sigma}^i_a = \partial_j\widetilde{\sigma}^k_a + f_{abc}A^b_j\widetilde{\sigma}^k_c = 0, \tag{8}$$

known as the Gauss' law constraint G_a, where f_{abc} are the $SO(3)$ structure constants corresponding to the adjoint representation of $SU(2)$. Equations (6) would correlate to (5) upon the implementation of reality conditions, but (8) imposes three additional constraints on the Ashtekar phase space $\Omega_{Ash} = (\widetilde{\sigma}^i_a, A^a_i)$. Since $Dim(\Omega_{EH}) = 6$ per point at the unconstrained level, then this begs the question as to the role of (8) in the original metric description. The Ashtekar unconstrained phase space has $Dim(\Omega_{Ash}) = 18$ complex degrees of freedom per point. Clearly, the implementation of (8) would reduce degrees of freedom from $Dim(\Omega_{Ash}) = 18$ to $Dim(\Omega_{Ash}) = 12$. Then the implementation of reality conditions would yield $Dim(\Omega_{Ash}) = 6$ which matches $Dim(\Omega_{EH})$ at the unconstrained level. Hence (8) subsequent to this phase space reduction cannot be allowed to impose any conditions on Ω_{EH}.

In this paper we will examine the metric representation of GR in terms of the instanton representation of Plebanski gravity. The instanton representation can most directly be obtained from the Ashtekar variables by using the CDJ Ansatz

$$\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e, \tag{9}$$

where $\Psi_{ae} \in SU(2)_- \otimes SU(2)_-$. Then the initial value constraints (6) and (8) can be written as constraints directly on the CDJ matrix Ψ_{ae}, given by

$$H_i = \epsilon_{ijk}B^j_aB^k_a\Psi_{ae}; \quad G_a = B^i_eD_i\Psi_{ae} \equiv \mathbf{w}_e\{\Psi_{ae}\} = 0; \quad H = \Lambda + \mathrm{tr}\Psi^{-1} = 0 \tag{10}$$

for the diffeomorphism, Gauss' law and Hamiltonian constraints respectively. In re-establishing the link to the metric theory in what follows, we will derive Einstein's gravity using the Gauss' law constraint (8) as a starting point.

3 Riemannian structure of the Ashtekar variables

3.1 The three dimensional affine connection

To elucidate the role of the Gauss' law constraint in metric GR, we will now investigate the Riemannian structure implied by the Ashtekar variables. Using a gauge connection A^a_i, define an affine connection Γ^k_{jm} for the uncontracted form of the $SU(2)_-$ gauge covariant derivative D_i, such that

$$D_i\widetilde{\sigma}^k_a = -\Gamma^k_{ij}\widetilde{\sigma}^j_a. \tag{11}$$

The condition $D_k\widetilde{\sigma}^k_a = 0$ is the Gauss' Law constraint G_a from (8). Let us take the trace of (11), except keeping the constraint explicit.

$$D_k\widetilde{\sigma}^k_a = G_a = -\Gamma^k_{kj}\widetilde{\sigma}^j_a \longrightarrow \Gamma^k_{kj} = -(\widetilde{\sigma}^{-1})^a_j G_a. \tag{12}$$

Let us now write (11) and (12) in terms of the triadic geometry of e^a_i. Assuming the invertibility of the densitized triad $\widetilde{\sigma}^i_a$, (11) yields

$$\Gamma^k_{ij} = -(\widetilde{\sigma}^{-1})^a_j D_i\widetilde{\sigma}^k_a. \tag{13}$$

Now expand (13) in terms of triad variables using (168)

$$\begin{aligned}\Gamma^k_{ij} &= -\frac{1}{2}(\det e)^{-1}e^a_j D_i(\epsilon^{kmn}\epsilon_{abc}e^b_m e^c_n)\\ &= -(\det e)^{-1}\epsilon^{kmn}(\epsilon_{abc}e^a_j e^b_m)D_i e^c_n = -(\det e)^{-1}(\det e)\epsilon^{kmn}\epsilon_{jml}E^l_c D_i e^c_n\\ &= -\big(\delta^k_j\delta^n_l - \delta^k_l\delta^n_j\big)E^l_c D_i e^c_n = E^k_c D_i e^c_j - \delta^k_j E^n_d D_i e^d_n.\end{aligned} \tag{14}$$

Similarly, (12) in the triad variables is given by

$$\Gamma^k_{kj} = -(\det e)^{-1}e^a_j G_a. \tag{15}$$

Taking the trace of (14) by summation over $i = k$, we obtain

$$\Gamma^k_{kj} = E^k_c D_k e^c_j - E^n_d D_j e^d_n. \tag{16}$$

Taking the trace of (11) by summation over $j = k$, we obtain

$$\Gamma^k_{ik} = E^k_c D_i e^c_k - 3E^n_d D_i e^d_n = -2E^n_d D_i e^d_n \longrightarrow E^n_d D_i e^d_n = -\frac{1}{2}\Gamma^k_{ik}. \tag{17}$$

Substitution of (17) back into (14) yields the relation

$$\Gamma^k_{ij} = E^k_c D_i e^c_j - \delta^k_j (E^n_d D_i e^d_n) = E^k_c D_i e^c_j + \frac{1}{2}\delta^k_j \Gamma^m_{im}$$
$$\longrightarrow E^k_c D_i e^c_j - \Gamma^k_{ij} = -\frac{1}{2}\delta^k_j \Gamma^m_{im}. \tag{18}$$

Multiplying (18) by e^a_k we have

$$D_i e^a_j - \Gamma^k_{ij} e^a_k = -\frac{1}{2}\Gamma^k_{ik} e^a_j. \tag{19}$$

We will now use (19) to determine the criterion for metricity of the affine connection with respect to the 3-metric $h_{ij} = e^a_i e^a_j$ constructed from the triad. Multiply (19) by e^a_m

$$e^a_m D_i e^a_j - \Gamma^k_{ij} e^a_k e^a_m = -\frac{1}{2}\Gamma^k_{ik} e^a_j e^a_m, \tag{20}$$

and write down the clone with m and j interchanged,

$$e^a_j D_i e^a_m - \Gamma^k_{im} e^a_k e^a_j = -\frac{1}{2}\Gamma^k_{ik} e^a_m e^a_j. \tag{21}$$

We will now add (20) and (21), but first let us apply the Leibniz rule to the terms containing gauge covariant covariant derivatives

$$e^a_m D_i e^a_j + e^a_j D_i e^a_m = D_i (e^a_m e^a_j) = D_i h_{mj} = \partial_i h_{mj}. \tag{22}$$

In (22) we have used the fact that the metric h_{ij} is invariant under internal rotations since it does not have any internal indices, which makes its gauge covariant derivative the same as the regular partial derivative. Hence, applying (22) to the sum of (20) and (21), we obtain

$$\partial_i h_{mj} - \Gamma^k_{ij} h_{km} - \Gamma^k_{im} h_{kj} = \nabla_i h_{mj} = -\Gamma^k_{ik} h_{mj}. \tag{23}$$

We have identified the left hand side of (23) as the covariant derivative of the 3-metric h_{ij}, seen as a covariant 3-tensor of second rank, with respect to the connection Γ^k_{im}. In a metric theory of gravity, the metric should be annihilated by the Levi–Civita covariant derivative ∇_i. The fact that the right hand side of (23) is nonzero indicates that Γ^k_{ij} is incompatible with h_{ij} by terms proportional to the Ashtekar Gauss' law constraint G_a and to torsion. Otherwise, the metric h_{ij} would satisfy the Einstein equation for the 3-dimensional Riemannian curvature corresponding to this connection.

Let us nevertheless compute the permutations of (23)

$$\begin{aligned}
&\partial_i h_{mj} - \Gamma^k_{ij} h_{km} - \Gamma^k_{im} h_{kj} = -\Gamma^k_{ik} h_{mj};\\
&-\partial_m h_{ji} + \Gamma^k_{mi} h_{kj} + \Gamma^k_{mj} h_{ki} = \Gamma^k_{mk} h_{ji};\\
&\partial_j h_{im} - \Gamma^k_{jm} h_{ki} - \Gamma^k_{ji} h_{km} = -\Gamma^k_{jk} h_{im}.
\end{aligned} \tag{24}$$

Adding the three lines of (24) and dividing by 2 we obtain

$$\begin{aligned}
\Gamma^k_{(ij)} h_{km} - \frac{1}{2}\big(\partial_i h_{mj} + \partial_j h_{mi} - \partial_m h_{ij}\big) &= \Gamma^k_{[im]} h_{kj} + \Gamma^k_{[jm]} h_{ik}\\
&\quad -\frac{1}{2}\big(\delta^n_i h_{mj} + \delta^n_j h_{im} - \delta^n_m h_{ij}\big)\Gamma^k_{nk}.
\end{aligned} \tag{25}$$

The right hand side of (25) contains contributions due to torsion $T^k_{ij} = \Gamma^k_{[ij]}$, and due to Γ^k_{ki}. There are two conditions necessary for Γ^k_{ij} to be uniquely determined by the 3-metric h_{ij}: (i) $\Gamma^k_{kn} = 0 \rightarrow G_a = 0$, which means that the Gauss' law constraint must be satisfied by the Ashtekar variables. (ii) Secondly, the connection Γ^k_{ij} must be torsion-free, namely that $\Gamma^k_{[ij]} = 0$. This means that

$$\Gamma^k_{[ij]} e^a_k = D_{[i} e^a_{j]} = \partial_i e^a_j - \partial_j e^a_i + f^{abc} A^b_i e^c_j - f^{abc} A^b_j e^c_i = 0. \tag{26}$$

Equation (26) has been written in terms of the gauge covariant derivative. The combination of (i) and (ii) make the right hand side of (25) vanish, yielding

$$\Gamma^k_{(ij)} = \frac{1}{2}\big(\partial_i h_{mj} + \partial_j h_{mi} - \partial_m h_{ij}\big). \tag{27}$$

Equation (26) implies that Γ^k_{ij} is torsion free, hence by (27) is the unique Levi–Civita connection compatible with the 3-metric h_{ij} derived from the triad e^a_i. In order for Γ^k_{ij} to be a metric connection, it was necessary to impose $G_a = 0$, which implies the invariance of h_{ij} under internal rotations. We will ultimately see that equation (26) must be relaxed to allow the theory to have torsion. This is because, as we will prove, torsion is necessary in order for the intrinsic 3-geometry to imply a four dimensional one.

3.2 The three dimensional Riemannian curvature

Let us now compute the three dimensional curvature, using what we know about the gauge theory. Write (19) in the form

$$D_j e^a_m = \Gamma^k_{jm} e^a_k + Q^a_{jm}, \tag{28}$$

where we have defined the nonmetricity $Q^a_{ij} = -\frac{1}{2}\Gamma^k_{ik} e^a_j$.[3] Taking the second gauge covariant derivative of (28) we have

$$\begin{aligned} D_i D_j e^a_m &= D_i(\Gamma^k_{jm} e^a_k) + D_i Q^a_{jm} \\ &= (\partial_i \Gamma^k_{jm}) e^a_k + \Gamma^k_{jm} D_i e^a_k + D_i Q^a_{jm}. \end{aligned} \tag{29}$$

We have used the fact that Γ^k_{ij} is a scalar under internal rotations since it does not contain any internal indices, which allows the replacement $D_i \rightarrow \partial_i$ to be made when acting on Γ^k_{ij}. Putting (28) into (29), we have

$$\begin{aligned} D_i D_j e^a_m &= (\partial_i \Gamma^k_{jm}) e^a_k + \Gamma^k_{jm}\big(\Gamma^n_{ik} e^a_n + Q^a_{ik}\big) + D_i Q^a_{jm} \\ &= \big(\partial_i \Gamma^n_{jm} + \Gamma^k_{jm}\Gamma^n_{ik}\big) e^a_n + D_i Q^a_{jm} + \Gamma^k_{jm} Q^a_{ik}. \end{aligned} \tag{30}$$

Equation (30) with i and j interchanged is given by

$$D_j D_i e^a_m = \big(\partial_j \Gamma^n_{im} + \Gamma^k_{im}\Gamma^n_{jk}\big) e^a_n + D_j Q^a_{im} + \Gamma^k_{im} Q^a_{jk}. \tag{31}$$

Subtracting (31) from (30) to commute the covariant derivatives, we have

$$\begin{aligned} [D_i, D_j] e^a_m = R^k_{mij} e^a_k = \Big(\partial_i \Gamma^n_{jm} - \partial_j \Gamma^n_{im} + \Gamma^k_{jm}\Gamma^n_{ik} - \Gamma^k_{im}\Gamma^n_{jk}\Big) e^a_n \\ + \overline{\nabla}_i Q^a_{jm} - \overline{\nabla}_j Q^a_{im}, \end{aligned} \tag{32}$$

where we have defined

$$\overline{\nabla}_i Q^a_{jm} = D_i Q^a_{jm} + \Gamma^k_{jm} Q^a_{ik} \equiv \boldsymbol{T}^a_{ijm}. \tag{33}$$

Rewriting the left hand side of (32) in terms of the gauge covariant derivative we have

$$[D_i, D_j] e^a_m = \epsilon_{ijk} f^{abc} B^k_b e^c_m = R^k_{mij} e^a_k, \tag{34}$$

where $B^i_a = \frac{1}{2}\epsilon^{ijk} F^a_{jk}$ is the magnetic field derived from $F^a_{jk}[A]$, the field strength of the connection A^a_i, and where the three dimensional Riemannian curvature is given by

[3] Recall that the nonmetricity is determined by the Gauss' law constraint G_a. We will retain this term in the computation to maintain generality, understanding that it is to be set to zero in order to obtain the original metric theory.

$$R^k_{mij}e^a_k=\Big(\partial_i\Gamma^n_{jm}-\partial_j\Gamma^n_{im}+\Gamma^k_{jm}\Gamma^n_{ik}-\Gamma^k_{im}\Gamma^n_{jk}\Big)e^a_n+\boldsymbol{T}^a_{[ij]m}. \tag{35}$$

Equation (34) relates the gauge curvature of the gauge connection A^a_i on the one hand, to a purely spatial 3-geometry on the other. Note that this 3-geometry off-shell may in general include the nonmetricity as well as torsion. Multiplying (34) by e^a_n we obtain

$$\epsilon_{ijk}f^{abc}B^k_b e^c_m e^a_n=R^k_{mij}e^a_k e^a_n, \tag{36}$$

which upon interchanging indices $m\leftrightarrow n$ is the same as

$$R^k_{nij}h_{km}=\epsilon_{ijk}f^{cab}e^c_n e^a_m B^k_b. \tag{37}$$

Since the left hand side of (37) is expressed entirely in terms of a 3-geometry, then we can lower the upper index on the curvature. We can also re-write the right hand side in terms of the densitized triad rewriting (168) as $\epsilon_{nml}\widetilde{\sigma}^l_b=\epsilon_{bca}e^c_n e^a_m$. The result of this is to transform (37) into the form

$$R_{mnij}=\epsilon_{mnl}\epsilon_{ijk}\widetilde{\sigma}^l_b B^k_b. \tag{38}$$

3.3 The CDJ Ansatz

Let us rewrite (38) for completeness, relabelling the indices

$$R_{ijmn}=\epsilon_{ijl}\epsilon_{mnk}\widetilde{\sigma}^l_b B^k_b. \tag{39}$$

On the one hand (39) is a gauge curvature for the Ashtekar connection A^a_i, and on the other hand it is also the purely spatial 3-dimensional Riemannian curvature tensor for the affine connection Γ^i_{jk}. The Ashtekar self-dual connection, being the pullback of the four dimensional spin connection to 3-space Σ, contains information about spacetime M. To make more concrete the link from 3-space Σ to a four dimensional geometry, let us first start by postulating the following relation

$$\widetilde{\sigma}^i_a=\Psi_{ae}B^i_e, \tag{40}$$

where Ψ_{ae} is a nondegenerate $SO(3,C)_-\otimes SO(3,C)_-$-valued matrix.[4] Let us eliminate B^i_a by substituting (40) into (39)

[4] This is known as the CDJ matrix, the traceless part of whose inverse was introduced in [7] to solve the Hamiltoninan and diffeomorphism constraints H_μ in the Ashtekar variables.

$$R_{ijmn}=\epsilon_{ijl}\epsilon_{mnk}\widetilde{\sigma}^l_f\widetilde{\sigma}^k_b\Psi^{-1}_{bf}. \tag{41}$$

Observe that the following can also be written as a consequence of (40), when Ψ_{ae} is nondegenerate as a three by three matrix

$$R_{ijmn}=\epsilon_{ijl}\epsilon_{mnk}\Psi_{bf}B^l_fB^k_b. \tag{42}$$

Taking the average of (41) and (42), we have $R_{ijmn}=\epsilon_{ijl}\epsilon_{mnk}T^{ij}$, where

$$T^{lk}=\frac{1}{2}\big(\Psi^{-1}_{bf}\widetilde{\sigma}^l_f\widetilde{\sigma}^k_b+\Psi_{bf}B^l_fB^k_b\big). \tag{43}$$

If we interpret $\widetilde{\sigma}^i_a$ and B^i_a respectively as the electric and the magnetic fields for a $SO(3,C)$ Yang–Mills theory, then equation (43) would be the spatial part of the stress energy tensor T^{ij} for this theory, where Ψ_{ae} is the gauge coupling constant. Hence one would expect T^{ij} to be symmetric and conserved, a point which we will return to shortly.

Starting with (41), which is the full 3-dimensional Riemannian curvature including nonmetricity and torsion, let us form the 3-dimensional Ricci tensor by contraction with h^{im}

$$\begin{aligned}h^{im}R_{ijmn}\equiv R_{jn}&=\epsilon_{ijl}\epsilon_{mnk}E^i_gE^m_g(E^l_fE^k_b)(\mathrm{det}e)^2\Psi^{-1}_{bf}\\=(\epsilon_{jli}E^l_fE^i_g)(\epsilon_{nkm}E^k_bE^m_g)(\mathrm{det}e)^2\Psi^{-1}_{bf}&=\epsilon_{fgd}\epsilon_{bge}e^d_je^e_n\Psi^{-1}_{bf}\\=\big(\delta_{fb}e^d_je^d_n-e^b_je^f_n\big)\Psi^{-1}_{bf}&=\big(\delta_{fb}h_{jn}-e^b_je^f_n\big)\Psi^{-1}_{bf}.\end{aligned} \tag{44}$$

The result is that

$$R_{jn}=h_{jn}\mathrm{tr}\Psi^{-1}-e^b_je^f_n\Psi^{-1}_{bf}. \tag{45}$$

Let us now obtain the 3-dimensional Ricci scalar by contracting (45) again

$${}^{(3)}R=h^{jn}h_{jn}\mathrm{tr}\Psi^{-1}-(E^j_aE^n_a)e^b_je^f_n\Psi^{-1}_{bf}=2\mathrm{tr}\Psi^{-1}. \tag{46}$$

Hence, putting $\mathrm{tr}\Psi^{-1}=\frac{1}{2}{}^{(3)}R$ into (45), we obtain

$$e^b_je^f_n\Psi^{-1}_{bf}=-\big(R_{jn}-\frac{1}{2}h_{jn}R\big)=-G_{jn}. \tag{47}$$

This provides the physical interpretation of the inverse CDJ matrix as the expression of the Einstein tensor G_{ij} for a 3-dimensional space containing nonmetricity and torsion, in the triad frame.

The CDJ matrix satisfies the relation

$$\Psi^{-1}_{bf}=-\frac{\Lambda}{3}\delta_{bf}+\psi_{bf}, \tag{48}$$

where ψ_{bf} is the antiself-dual part of the Weyl curvature, which is symmetric and traceless. For $\psi_{bf}=0$, conformally flat space, (47) reduces to

$$G_{jn}=\frac{\Lambda}{3}h_{jn} \tag{49}$$

which in the language of [1] corresponds to the 3-geometry of an Einstein space. For $\psi_{ae}\neq 0$ one should expect to obtain a more general space which incorporates the degrees of freedom of general relativity.

3.4 Interpretation of the initial value constraints

Let us now see what the initial value constraints on Ψ_{ae} imply about the 3-dimensional spatial geometry, by expressing (10) in terms of G_{ij}. First, note that a necessary condition for G_{ij} to be symmetric in i,j is that Ψ_{ae} be symmetric in a,e. This latter condition can be imposed by implementation of the diffeomorphism constraint using (40), whence

$$H_i=\epsilon_{ijk}B^j_aB^k_e\Psi_{ae}=\epsilon_{ijk}\widetilde{\sigma}^j_a\widetilde{\sigma}^k_e\Psi^{-1}_{ae}=0. \tag{50}$$

For the Einstein tensor G_{ij} to arise from a metric theory we must impose the Gauss' law constraint, which in terms of the CDJ matrix is given by

$$G_a=B^i_eD_i\Psi_{ae}=\mathbf{w}_e\{\Psi_{ae}\}=0, \tag{51}$$

where we have used the Gauge Bianchi identity $D_iB^i_e=0$. We will now show that the Gauss' law constraint (51) implies that G_{ij} satisfies a spatial Bianchi identity. First let us act on both sides of (47) with the gauge covariant derivative

$$-D_kG_{ij}=D_k(\Psi^{-1}_{bf}e^b_ie^f_j), \tag{52}$$

making use of (19). We can replace the action on the left hand side of (52) with the partial derivative, since G_{ij} does not have internal indices. Hence using (19), we have

$$
\begin{aligned}
-\partial_k G_{ij} &= (D_k \Psi^{-1}_{bf}) e^b_i e^f_j + \Psi^{-1}_{bf} (D_k e^b_i) e^f_j + \Psi^{-1}_{bf} e^b_i (D_k e^f_j) \\
= -\Psi^{-1}_{bg} (D_k \Psi_{gd}) \Psi^{-1}_{df} e^b_i e^f_j &+ \Psi^{-1}_{bf} \big(\Gamma^m_{ki} e^b_m - \frac{1}{2} \Gamma^m_{km} e^b_i\big) e^f_j + \Psi^{-1}_{bf} \big(\Gamma^m_{kj} e^f_m - \frac{1}{2} \Gamma^m_{km} e^f_j\big) e^b_i \\
= -\Psi^{-1}_{bg} (D_k \Psi_{gd}) \Psi^{-1}_{df} e^b_i e^f_j &+ \Psi^{-1}_{bf} \Gamma^m_{ki} e^b_m e^f_j + \Psi^{-1}_{bf} \Gamma^m_{kj} e^f_m e^b_i - \Psi^{-1}_{bf} \Gamma^m_{km} e^b_i e^f_j . \quad (53)
\end{aligned}
$$

The last term on the right hand side of (53) is given by

$$
-\Psi^{-1}_{bf} \Gamma^m_{km} e^b_i e^f_j = \Gamma^m_{km} G_{ij}. \tag{54}
$$

Using (51) and subtracting the corresponding terms to the left hand side, we have

$$
\begin{aligned}
-\partial_k G_{ij} + \Gamma^m_{ki} G_{mj} + \Gamma^m_{kj} G_{im} &= -\nabla_k G_{ij} \\
&= -\Psi^{-1}_{bg} (D_k \Psi_{gd}) \Psi^{-1}_{df} e^b_i e^f_j + \Gamma^m_{km} G_{ij}.
\end{aligned} \tag{55}
$$

We recognize the left hand side as $\nabla_k G_{ij}$, the covariant derivative of G_{ij}, seen as a second rank tensor, with respect to the connection Γ^k_{ij}. To form the Bianchi identity, let us contract both sides of (55) with h^{kj}, hence

$$
-h^{kj} \nabla_k G_{ij} = -\nabla^k G_{ik} = -\Psi^{-1}_{bg} (D_k \Psi_{gd}) \Psi^{-1}_{df} E^k_f e^b_i + h^{kj} \Gamma^m_{km} G_{ij}. \tag{56}
$$

Applying the CDJ Ansatz to the following triadic object

$$
\Psi^{-1}_{df} E^k_f = \frac{\Psi^{-1}_{df} \widetilde{\sigma}^k_f}{\sqrt{\det \widetilde{\sigma}}} = B^k_d (\det e)^{-1}, \tag{57}
$$

and subsituting into (57) and defining $\mathbf{w}_d \equiv B^k_d D_k$, we have

$$
-\nabla^k G_{ik} = -(\det e)^{-1} \Psi^{-1}_{bg} \mathbf{w}_d \{\Psi_{gd}\} e^b_i + h^{kj} \Gamma^m_{km} G_{ij}. \tag{58}
$$

In the last term of (58) we have written the Gauss' law constraint in terms of the CDJ matrix. Setting $\mathbf{w}_d \{\Psi_{gd}\} = 0$ implies that $\nabla^k G_{ik} = 0$ when Γ^k_{ij} is torsion-free. Hence the Einstein tensor G_{ij} violates the Bianchi identity to the extent that there is torsion in the theory.

Lastly, the Hamiltonain constraint on the CDJ matrix is given by

$$
H = \mathrm{tr} \Psi^{-1} + \Lambda = 0, \tag{59}
$$

which in terms of G_{ij} is given by

$$-h^{ij}G_{ij} + \Lambda = \frac{1}{2}{}^{(3)}R + \Lambda = 0. \tag{60}$$

This implies that ${}^{(3)}R = -2\Lambda$ on-shell, where ${}^{(3)}R$ is the Riemann curvature scalar for a three dimensional space having torsion. Returning to (43), we see that the identification $G^{ij} \sim T^{ij}$ as a Yang–Mills spatial energy momentum tensor is true when the initial value constraints of GR are satisfied. Moreover, one sees from (60) that the trace of this energy momentum tensor is the cosmological constant Λ. The energy momentum tensor violates the covariant conservation law by terms proportional to the torsion of this 3-space.

3.5 The gauge connection

Let us find out more about the gauge connection as defined in (8). Writing the Gauss' law constraint in triad variables we have

$$G_a = D_i\widetilde{\sigma}^i_a = \frac{1}{2}D_i(\epsilon^{ijk}\epsilon_{abc}e^b_je^c_k) = \epsilon^{ijk}\epsilon_{abc}e^b_jD_ie^c_k. \tag{61}$$

Expanding the gauge covariant derivative, we have

$$G_a = \epsilon^{ijk}\epsilon_{abc}e^b_j\big(\partial_ie^c_k + f^{cfg}A^f_ie^g_k\big). \tag{62}$$

Now make the following split of the gauge connection

$$A^f_i = \Gamma^f_i + \beta K^f_i, \tag{63}$$

where β is a numerical constant, and Γ^a_i and K^a_i remain to be determined. Putting (63) into (62), we have

$$G_a = \epsilon^{ijk}\epsilon_{abc}e^b_j\big(\partial_ie^c_k + f^{cfg}\Gamma^f_ie^g_k\big) + \beta\epsilon^{ijk}\epsilon_{abc}e^b_jf^{cfg}K^f_ie^g_k. \tag{64}$$

We will now impose the condition that Γ^a_i be the spin connection compatible with the triad e^a_i, namely that

$$\epsilon^{ijk}\overline{D}_je^c_k = \epsilon^{ijk}\big(\partial_je^c_k + f^{cfg}\Gamma^f_je^g_k\big) = 0. \tag{65}$$

Equation (65) causes the first term on the right hand side of (64) to vanish, leaving the second term. Let us now simplify this term

$$G_a = \beta(\epsilon_{abc}\epsilon^{fgc})(\epsilon^{ijk}e^b_j e^g_k)K^f_i$$
$$= \beta\big(\delta^f_a\delta^g_b - \delta^f_b\delta^g_a\big)(\mathrm{det}e)\epsilon^{bgd}E^i_d K^f_i = -\beta(\mathrm{det}e)\epsilon^{fad}E^i_d K^f_i. \quad (66)$$

Let us make the definition

$$K^a_k = e^a_j K^j_k, \quad (67)$$

where K^j_k is purely spatial. We will now contract both sides of (67) with e^a_i and make use of the Riemannian structure

$$K^a_k e^a_l = e^a_j e^a_l K^j_k = h_{jl}K^j_k \equiv K_{lk}, \quad (68)$$

whence the upper index of K^j_i becomes lowered by the 3-metric h_{ij}. Contraction of (68) with E^l_b implies

$$K^b_k = E^l_b K_{lk}, \quad (69)$$

with signifies that for internal indices, the upper and lower index positions are equivalent, but for spatial indices they are not equivalent. Using (69), we can further write (66) as

$$G_a = -\beta(\mathrm{det}e)\epsilon^{fad}E^i_d E^j_f K_{ji} = -\beta\epsilon^{ijk}K_{ji}e^a_k. \quad (70)$$

Therefore when the Gauss' law constraint is satisfied in the Ashtekar variables, this means that for all Γ^a_i chosen to satisfy (65), K_{ij} must necesarily be symmetric in i and j. The converse of this is also true.

A counting argument demands, as a requirement of consistency, that the the connection Γ^i_{jk} must be allowed to have torsion. This can be seen by writing (63) as

$$A^a_i = \Gamma^a_i[e] + \beta E^j_a K_{ij}. \quad (71)$$

Since from (65), Γ^a_i is the torsion-free spin connection compatible with e^a_i, then it is completely determined by e^a_i. Since e^a_i has nine degrees of freedom, then it follows that Γ^a_i must also have nine degrees of freedom. Since the vanishing of (70) implies $K_{ij} = K_{(ij)}$, then K_{ij} has six degrees of freedom. Therefore on-shell, A^a_i has $9 + 6 = 15$ real degrees of freedom.[5] The decomposition of the affine connection Γ^i_{jk} is given by

[5]More precisely, since A^a_i contains 18 real components, it is a fifteen dimensional manifold embedded in an 18 dimensional space.

$$\Gamma^i_{jk} = \Gamma^i_{(jk)}[h] + \Gamma^i_{[jk]}. \tag{72}$$

The symmetric part of (72) contains six rather than 18 degrees of freedom, since it is determined entirely by the 3-metric h_{ij} (namely, it is a six dimensional manifold embedded in 18 dimensional space). The antisymmetric part, which is the torsion, contains nine degrees of freedom. This yields a total of fifteen D.O.F., which is consistent with those of A^a_i. Indeed, Γ^a_i may be identified with the connection $\Gamma^k_{(ij)}$ which contains six degrees of freedom. The additional three degrees of freedom inherent in Γ^a_i may be attributed to the freedom to perform $SO(3,C)$ rotations of the internal index a. This implies that six out of the nine degrees of freedom in the torsion $\Gamma^j_{[jk]}$ can be correlated to the extrinsic curvature K_{ij}.

4 Four dimensional curvature

Equation (41), repeated here

$$R_{ijmn} = \epsilon_{ijl}\epsilon_{mnk}\widetilde{\sigma}^l_f\widetilde{\sigma}^k_b\Psi^{-1}_{bf} = -\epsilon_{ijl}\epsilon_{mnk}G^{kl}h, \tag{73}$$

as well as (40) can be seen as the defining relation for Ψ_{ae}. Though (73) appears as a purely spatial object with no apparent reference to a four dimensional geometry, we will show that it actually corresponds to a four dimensional theory of gravitation. Identify $\widetilde{\sigma}^i_a$ with the spatial part of a self dual triple of two forms $\widetilde{\sigma}^a$, where

$$\widetilde{\sigma}^a = \Sigma^a_{ij}dx^i \wedge dx^j = \epsilon_{ijk}\widetilde{\sigma}^k_a dx^i \wedge dx^j, \tag{74}$$

with Σ^a given by (167). Then

$$\widetilde{\sigma}^a = \Sigma^a\Big|_{\vec{N}=\vec{\beta}=N=0}, \tag{75}$$

which allows (41) to be written as

$$R_{ijmn} = \Sigma^b_{ij}\Sigma^f_{mn}\Psi^{-1}_{bf}. \tag{76}$$

Contraction of (76) yields

$$R_{ijmn}dx^i \wedge dx^j \wedge dx^m \wedge dx^m = 0, \tag{77}$$

which vanishes since we have attempted to construct a four form on a three dimensional space Σ. Then the absence of a manifest four dimensional geometry in (76) becomes clear from the fact that Σ^a_{0i}, the temporal components of this two form Σ^a are explicitly absent from (77). Since from (178) and (179) Σ^a_{0i} contain $N^\mu = (N, N^i)$, the lapse and shift functions needed to implement (6), this suggests that (76) might be equivalent to four dimensional GR when the initial value constraints are satisfied. Moreover, this solution must be encoded within Ψ^{-1}_{bf}, or alternatively in G_{ij}.

To show this in a more rigorous way let us expand the left hand side of (76), which is a 3-dimensional curvature tensor of a metric space with torsion.[6] So let us first make the split

$$\Gamma^k_{ij} = \Gamma^k_{(ij)}[h] + \Gamma^k_{[ij]} = \Gamma^k_{(ij)} + T^k_{ij}. \tag{78}$$

[6] We have set the nonmetricity equal to zero.

The connection Γ^k_{ij} contains fifteen degrees of freedom. There are six D.O.F. in the symmetric part $\Gamma^k_{(ij)}$, since it is determined completely by the 3-metric $h_{ij} = e^a_i e^a_j$. There are nine D.O.F. in the torsion term $T^k_{ij} = \Gamma^k_{[ij]}$, namely three D.O.F. in the upper index and three D.O.F. in the antisymmetric combination of lower indices. So we must show that this correlates to a four dimensional theory of gravitation containing the same number of degrees of freedom. But first, let us expand out the curvature. This is given by

$$R^n_{ijm} = \partial_i \Gamma^n_{jm} - \partial_j \Gamma^n_{im} + \Gamma^k_{jm}\Gamma^n_{ik} - \Gamma^k_{im}\Gamma^n_{jk} \tag{79}$$

where we have set the nonmetricity contribution to (30) to zero. Substituting (78) into (79) we have

$$\begin{aligned} R^n_{ijm} &= \partial_i(\Gamma^n_{(jm)} + T^n_{jm}) - \partial_j(\Gamma^n_{(im)} + T^n_{im}) \\ &+(\Gamma^k_{(jm)} + T^k_{jm})(\Gamma^n_{(ik)} + T^n_{ik}) - (\Gamma^k_{(im)} + T^k_{im})(\Gamma^n_{(jk)} + T^n_{jk}) \\ &= {}^{(3)}R^n_{ijm}[h] + T^k_{jm}T^n_{ik} - T^k_{im}T^n_{jk} + \nabla_{[i}T^n_{j]m}, \end{aligned} \tag{80}$$

where we have defined

$${}^{(3)}R^n_{ijm}[h] = \partial_i \Gamma^n_{(jm)} - \partial_j \Gamma^n_{(im)} + \Gamma^k_{(jm)}\Gamma^n_{(ik)} - \Gamma^k_{(im)}\Gamma^n_{(jk)} \tag{81}$$

which is the three dimensional intrinsic curvature of 3-space Σ based on the 3-metric h_{ij}. Additionally, we have extracted the covariant derivative of the torsion T^k_{ij} as a third rank mixed tensor, by adding and subtracting the quantity $\Gamma^k_{(ij)}T^n_{km}$. This is given by

$$\nabla_i T^n_{jm} = \partial_i T^n_{jm} - \Gamma^k_{(im)}T^n_{jk} + \Gamma^n_{(ik)}T^k_{jm} - \Gamma^k_{(ij)}T^n_{km} \tag{82}$$

It is tempting to attempt to identify the quadratic torsion terms with extrinsic curvature for Einstein's general relativity, which would make (80) resemble the 3+1 decomposition of a four dimensional curvature.

Contract n with j in (80) to form the 3-dimensional Ricci tensor

$$R_{im} = R^j_{ijm} = {}^{(3)}R_{im}[h] + T^k_{jm}T^j_{ik} - T^k_{im}T^j_{jk} + \nabla_{[i}T^j_{j]m}. \tag{83}$$

Then contract (83) with h^{im} to form the 3-dimensional Ricci scalar

$$R = h^{im}R_{im} = {}^{(3)}R[h] + h^{im}T^k_{jm}T^j_{ik} + h^{im}\nabla_{[i}T^j_{j]m}, \tag{84}$$

which eliminates one of the torsion squared terms due to antisymmetry. The remaining torsion squared term is of the form $T^k_{jm}T^{jm}_k$, which resembles an extrinsic curvature squared term, with the remaining three degrees of freedom used to absorb the covariant divergence term in (84). To verify this, decompose the torsion as follows[7]

$$T^n_{im} = \epsilon_{iml}K^{ln} + \frac{1}{2}\big(\delta^n_i a_m - \delta^n_m a_i\big), \tag{85}$$

where K^{ln} is symmetric. The physical interpretation of a_m arises from taking the trace of (85), which yields $a_m = T^i_{im}$. Substitution of (85) into (84) yields

$$h^{im}T^k_{jm}T^j_{ik} = h^{im}\epsilon_{ikn}\epsilon_{mlj}K^{kl}K^{nj} - \frac{1}{2}h^{ij}a_i a_j. \tag{86}$$

Upon making the identification

$$h^{im}\epsilon_{ikn}\epsilon_{mlj} = h_{kl}h_{nj} - h_{kj}h_{nl}, \tag{87}$$

the first term of (86) indeed reduces to $VarK$, which confirms the interpretation of K^{ij} as the extrinsic curvature tensor. This also implies that ϵ_{ijk} is a tensor density, corresponding to curved space. Substituting these results into (84) yields

$$R = {}^{(3)}R + (\mathrm{tr}K)^2 - K^{ij}K_{ij} + \nabla^m a_m - \frac{1}{2}a^m a_m. \tag{88}$$

The 3-vector a_m is directly proportional to the nonmetricity of the theory. Setting $a_m = 0$ enforces that the theory be metric, whereupon the right hand side becomes the 3+1 decomposition of the four dimensional Riemann curvature tensor. This implies that the right hand side is the four dimensional Riemann curvature tensor of Einstein's general relativity. Hence, we have obtained Einstein's GR in four dimensions, starting from a three dimensional intrinsic geometry implied by the instanton representation. This implies for a general four dimensional space that the decomposition of its three dimensional affine connection is given by

$$\Gamma^k_{ij} = \Gamma^k_{(ij)} + \epsilon_{ijm}K^{mk} + \frac{1}{2}\big(\delta^k_i\delta^n_j - \delta^k_j\delta^i_n\big)(\widetilde{\sigma}^{-1})^a_n G_a. \tag{89}$$

Equation (89), upon setting $G_a = 0$, is equivalent to the Ashtekar connection written in the language of spatial indices.

[7] This is the spatial analogue of the decomposition of the structure constants of a Lie algebra into Bianchi type, albeit in the full theory and not minisuperspace.

5 Transformation from the instanton representation into the metric representation

5.1 The canonical structure

We have shown the link from the instanton representation to an intrinsic 3-geometry, which suggests the existence of a 4-geometry. We will now solidify this link by demonstrating that the instanton representation can be transformed directly into the metric representation. First we will deal with the canonical one form, using the Ashtekar variables for expediency

$$\boldsymbol{\theta}_{Inst} = \Psi_{ae} B^i_e \dot{A}^a_i = \widetilde{\sigma}^i_a \dot{A}^a_i. \tag{90}$$

We would like to express (90) directly on the metric phase space, as follows. Using the Liebniz rule, write (90) in the form

$$\widetilde{\sigma}^i_a \dot{A}^a_i = \frac{d}{dt}(\widetilde{\sigma}^i_a A^a_i) - A^a_i \dot{\widetilde{\sigma}}^i_a. \tag{91}$$

The total time derivative term is the generator of a canonical transformation into the new phase space. Note in the second term of (91) that the time derivative acts on the densitized triad, which is more closely related to the 3-metric than is the connection. Next, expand the resulting form in terms of the triad

$$A^a_i \dot{\widetilde{\sigma}}^i_a = \frac{1}{2} A^a_i \frac{d}{dt}(\epsilon^{ijk}\epsilon_{abc} e^b_j e^c_k) = A^a_i \epsilon^{ijk}\epsilon_{abc} e^b_j \dot{e}^c_k. \tag{92}$$

Next, expand the gauge connection $A^a_i = \Gamma^a_i + \beta K^a_i$

$$\boldsymbol{\theta} = \epsilon^{ijk}\epsilon_{abc}\big(\Gamma^a_i e^b_j + \beta K^a_i e^b_j\big)\dot{e}^c_k. \tag{93}$$

Equation (93) splits into two terms. Let us first simplify the second term, proportional to β. This is given by

$$\begin{aligned} \beta\epsilon^{ijk}\epsilon_{abc} K^a_i e^b_j &= \beta\epsilon^{ijk}\epsilon_{abc} e^a_n K^n_i e^b_j \\ = \beta\epsilon^{ijk}(\text{dete})\epsilon_{njm} E^m_c K^n_i &= \beta(\text{dete})\big(\delta^i_n\delta^k_m - \delta^i_m\delta^k_n\big) E^m_c K^n_i \\ &= \beta(\text{dete})\big(E^k_c(\text{tr}K) - E^i_c K^k_i\big). \end{aligned} \tag{94}$$

Taking the time derivative of the following relation $e^c_k e^c_n = h_{kn}$,

$$\dot{h}_{kn} = 2e^c_n \dot{e}^c_k \longrightarrow \dot{e}^c_k = \frac{1}{2} E^n_c \dot{h}_{kn}, \tag{95}$$

we have upon substitution of (95) into (94) that

$$\beta(\det e)\big(E^k_c(\mathrm{tr}K) - E^i_c K^k_i\big) = \frac{\beta}{2}(\det e)\big(E^k_c E^n_c(\mathrm{tr}K) - E^i_c E^n_c K^k_i\big)\dot{h}_{kn}$$
$$= \frac{\beta}{2}\sqrt{h}\big(h^{kn}(\mathrm{tr}K) - K^{nk}\big)\dot{h}_{kn}, \quad (96)$$

where we have used $K^{nk} = h^{in}K^k_i$.

Define the coefficient of $\dot{h}_{kn}$ in (96) as $\pi^{ij} = \sqrt{h}(K^{ij} - h^{ij}(\mathrm{tr}K))$. Then the canonical one form (93) is given by

$$\boldsymbol{\theta} = \frac{\beta}{2}\pi^{kn}\dot{h}_{kn} + \epsilon^{ijk}\epsilon_{abc}\Gamma^a_i e^b_j \dot{e}^c_k. \quad (97)$$

Let us now evaluate the second term of (97). Recall the definition of Γ^a_i as the unique spin connection compatible with e^a_i, as in

$$\overline{D}_i e^b_j = 0 \longrightarrow f^{bcd}\Gamma^c_j e^d_j = -\partial_i e^b_j. \quad (98)$$

Substuting (98) into the second term of (97), we have

$$\epsilon^{ijk}\epsilon_{abc}\Gamma^a_i e^b_j \dot{e}^c_k = -\epsilon^{ijk}(\partial_i e^c_j)\dot{e}^c_k. \quad (99)$$

Equation (99) can be written, omitting the minus sign

$$\epsilon^{ijk}\dot{e}^c_k(\partial_i e^c_j) = \frac{d}{dt}(\epsilon^{kij}e^c_k(\partial_i e^b_j)) - \epsilon^{kij}e^c_k(\partial_i \dot{e}^c_j). \quad (100)$$

The first term of (100) is a total time derivative, which forms a contribution to the generator of a canonical transformation. We will now show that the second term vanishes. Integrating this by parts with respect to 3-space we have

$$-\epsilon^{kij}e^c_k(\partial_i \dot{e}^c_j) = -\partial_i(\epsilon^{kij}e^c_k \dot{e}^c_j) + \epsilon^{kij}(\partial_i e^c_k)\dot{e}^c_j. \quad (101)$$

The first term of (101) is a spatial boundary term, which we will require to vanish. Let us now manipulate the first term as follows

$$\epsilon^{kij}\dot{e}^c_j(\partial_i e^c_k) = \epsilon^{jik}\dot{e}^c_k(\partial_i e^c_j) = \epsilon^{kji}\dot{e}^c_k(\partial_i e^c_j) = -\epsilon^{kij}\dot{e}^c_k(\partial_i e^c_j) = 0. \quad (102)$$

We have proven, modulo boundary terms, that (99) is its own negative, hence it must be zero. This can also be seen using the coordinate-invariant notation of differential forms, whence (99) takes the form

$$d(e \wedge de) = de \wedge de, \tag{103}$$

which is a total derivative that integrates to a boundary term.

We must now tally all the total time derivative terms that were necessary in order to obtain this result. These appear in (91) and in (100), which provides the terms that must be added to the starting action

$$\int_\Sigma d^3x \big(\widetilde{\sigma}^i_a A^a_i - (\epsilon^{ijk} e^c_i (\partial_j e^b_k)) \big). \tag{104}$$

The result, from (96), is that the canonical one form transforms from the Ashtekar variables into

$$\boldsymbol{\theta} = \frac{\beta}{2} \int dt \int_\Sigma d^3x \pi^{ij} \dot{h}_{ij}. \tag{105}$$

There are three remaining tasks from (105). The first task is to obtain the diffeomorphism constraint. This is obtained by generalizing the time derivative of h_{ij} to its Lie derivative along the vector normal to Σ, which should induce the diffeomorphism constraint. The second task is to perform the Legendre transformation induced by (105) into the Hamiltonian description. For the first task, note that the Lie derivative of the spacetime metric $g_{\mu\nu}$ along the vector field ξ^ν is given by

$$L_\xi g_{\mu\nu} = \xi^\rho \partial_\rho g_{\mu\nu} - \xi^\rho_{,\mu} g_{\rho\nu} - \xi^\rho_{,\nu} g_{\mu\rho}. \tag{106}$$

Choosing $\xi^\nu = \delta^\nu_0$ to be perpendicular to Σ, then (106) reduces to

$$L_\xi g_{\mu\nu} = \dot{g}_{\mu\nu} - \nabla_\mu g_{0\nu} - \nabla_\nu g_{0\mu}, \tag{107}$$

where ∇_ν is the four dimensional Levi-Civita connection compatible with $g_{\mu\nu}$. Defining the shift vector as $N_i = g_{0i}$ and taking the spatial components of (107), we obtain

$$L_\xi h_{ij} = \dot{h}_{ij} - \nabla_i N_j - \nabla_j N_i, \tag{108}$$

Multiplying (108) by the conjugate momentum π^{ij}, we obtain a contribution to the canonical one form which upon integration by parts equals

$$-2(\nabla_i N_j)\pi^{ij} \longrightarrow 2N_i \nabla_j \pi^{ij}, \tag{109}$$

the factor of 2 due to symmetry of π^{ij}. Hence the canonical one form transforms into

$$\frac{\beta}{2}\pi^{ij}L_{\xi}h_{ij} \sim \frac{\beta}{2}\pi^{ij}\dot{h}_{ij} + \beta N_i \nabla_j \pi^{ij}. \tag{110}$$

5.2 The Hamiltonian constraint

Next, we must transform the Hamiltonian constraint into the metric representation. The smeared Hamiltonian constraint for the instanton representation is given by

$$\boldsymbol{H}_{Inst} = \int dt \int_{\Sigma} d^3x N(\mathrm{det}B)^{1/2}\sqrt{\mathrm{det}\Psi}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{111}$$

The defining relation for the instanton representation is the CDJ Ansatz

$$\widetilde{\sigma}^i_a = \Psi_{ae}B^i_e, \tag{112}$$

where B^i_a is the Ashstekar magnetic field given by

$$B^i_a[A] = \epsilon^{ijk}\partial_j A^a_k + \frac{1}{2}\epsilon^{ijk}f_{abc}A^b_j A^c_k, \tag{113}$$

which is derived from the field strength of the Ashtekar gauge connection A^a_i. While (112) is valid only for nondegenerate B^i_a and nondegenerate Ψ_{ae}, there is no restriction on metric gravity due to the latter's being based on nondegenerate metrics $g_{\mu\nu}$. The determinant of (112) implies

$$N\sqrt{\mathrm{det}B}\sqrt{\mathrm{det}\Psi} = N\sqrt{\mathrm{det}\widetilde{\sigma}} = N\sqrt{h} = \sqrt{-g}, \tag{114}$$

which when substituted into (111) yields

$$\boldsymbol{H}_{Inst} = \int dt \int_{\Sigma} d^3x \sqrt{-g}\big(\Lambda + \mathrm{tr}\Psi^{-1}\big). \tag{115}$$

Next we will use the condition of nondegeneracy to invert (112) into the form

$$\Psi^{-1}_{ae} = B^i_e(\widetilde{\sigma}^{-1})^a_i, \tag{116}$$

then decompose the connection A^a_i as follows

$$A_i^a = \Gamma_i^a + \beta K_i^a, \tag{117}$$

where β is the Immirzi parameter. Recall that we have imposed the condition that the connection $\Gamma_i^a = \Gamma_i^a[e]$ is the unique torsion free spin connection compatible with some triad e_i^a via[8]

$$\epsilon^{ijk}\overline{D}_j e_k^a = \epsilon^{ijk}\big(\partial_j e_k^a + f^{abc}\Gamma_j^b e_k^c\big) = 0, \tag{118}$$

such that the Ashtekar densitized triad is given by $\widetilde{\sigma}_a^i = \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}e_j^b e_k^c$. Upon inserting (117) into (113) using (116), one obtains

$$\Psi_{ae}^{-1} = (\widetilde{\sigma}^{-1})_i^e\big({}^{(3)}R^{ai} + \frac{1}{2}\beta^2\epsilon^{ijk}f^{abc}K_j^b K_k^c + \beta\epsilon^{ijk}\overline{D}_j K_k^a\big). \tag{119}$$

In (119) we have defined

$${}^{(3)}R^{ai}[\Gamma] = \epsilon^{ijk}\big(\partial_j\Gamma_k^a + \frac{1}{2}f^{abc}\Gamma_j^b\Gamma_k^c\big), \tag{120}$$

which is the 3-dimensional curvature of Γ_i^a, and

$$\epsilon^{ijk}\overline{D}_j K_k^a = \epsilon^{ijk}\big(\partial_j K_k^a + f^{abc}\Gamma_j^b K_k^c\big), \tag{121}$$

which is the covariant curl of K_k^a with respect to Γ_i^a.

5.3 Evaluation of the terms

We will now evaluate each term of the trace of (119), according to its order in β. The zeroth order term of (119) can be written, using (171), in the form

$$(\widetilde{\sigma}^{-1})_i^a{}^{(3)}R^{ai}[\Gamma] = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}E_b^j E_c^k{}^{(3)}R^{ai}[\Gamma] \equiv \frac{1}{2}E_b^j E_c^k R_{jk}^{bc}[\Gamma] \tag{122}$$

where we have defined $R_{jk}^{bc} = R^{ai}\epsilon_{ijk}\epsilon^{abc}$. On account of the torsion-free condition (118), then Γ_i^a is the triadic form of the three dimensional Levi–Civita connection compatible with the 3-metric $h_{ij} = e_i^a e_j^a$. Hence (122) is also given by

[8]We have used the notation $\overline{D}_j$ to distinguish it from D_j, which refers to the full Ashtekar connection.

$$(\widetilde{\sigma}^{-1})^a_i {}^{(3)}R^{ai}[\Gamma[e]] = \frac{1}{2}{}^{(3)}R[h], \tag{123}$$

which is the intrinsic curvature of the 3-dimensional spatial hypersurface Σ. To compute the second term of (119), the term of order β^2, we need to derive the following preliminary result

$$\begin{aligned}\epsilon^{ijk} f^{abc}(\widetilde{\sigma}^{-1})^a_i K^b_j K^c_k &= \frac{1}{2}\epsilon_{imn}\epsilon^{afg}E^m_f E^n_g \epsilon^{ijk} f^{abc} K^b_j K^c_k \\ &= \frac{1}{2}\big(\delta^j_m\delta^k_n - \delta^j_n\delta^k_m\big)\big(\delta^{fb}\delta^{gc} - \delta^{fc}\delta^{gb}\big)E^m_f E^n_g K^b_j K^c_k \\ &= \frac{1}{2}\big(E^j_f E^k_g - E^j_g E^k_f\big)\big(K^f_j K^g_k - K^g_j K^f_k\big) \\ &= \big((E^j_g K^g_j)^2 - (E^j_g K^g_k)(E^j_f K^f_k)\big) = (\mathrm{tr}K)^2 - \mathrm{tr}K^2. \end{aligned} \tag{124}$$

In the last line of (124) we have used (69) to obtain

$$E^j_g K^g_j = E^j_g E^l_g K_{lj} = H^{lj} K_{lj} = \mathrm{tr}K. \tag{125}$$

So (124) will result in a contribution to $\mathrm{tr}\Psi^{-1}$ of

$$\frac{1}{2}\beta^2 \epsilon^{ijk} f^{abc} K^b_j K^c_k = \frac{1}{2}\beta^2\big((\mathrm{tr}K)^2 - K_{ij}K^{ij}\big). \tag{126}$$

where $Var = (\mathrm{tr}K)^2 - \mathrm{tr}K^2$ and we have used (67).

We will now compute the third term of (119), the term proportional to β. Upon substitution of (171), this yields

$$(\widetilde{\sigma}^{-1})^a_i \epsilon^{ijk}\overline{D}_j K^a_k = \frac{1}{2}\epsilon_{imn}\epsilon^{ijk}\epsilon^{afg}E^m_f E^n_g \overline{D}_j K^a_k = \epsilon^{afg}E^j_f E^k_g \overline{D}_j K^a_k. \tag{127}$$

In preparation for some future manipulations let us multiply (127) by $\sqrt{-g}$, using the following relation from (172),

$$\sqrt{-g} = N\sqrt{h} = N(\mathrm{det}e) = N(\mathrm{det}E)^{-1}. \tag{128}$$

Multiplying (127) by (128) and using (146), we have

$$\sqrt{-g}(\widetilde{\sigma}^{-1})^a_i \epsilon^{ijk}\overline{D}_j K^a_k = N(\mathrm{det}E)^{-1}\epsilon^{afg}E^j_f E^k_g \overline{D}_j K^a_k = N\epsilon^{ljk}e^a_l \overline{D}_j K^a_k. \tag{129}$$

Now use the Liebniz rule on (129) to transfer the derivative to e^a_l

$$N\epsilon^{ljk}e^a_l\overline{D}_jK^a_k = \partial_j(N\epsilon^{ljk}e^a_lK^a_k) - NK^a_k(\epsilon^{kjl}\overline{D}_je^a_l) - K^a_ke^a_l\epsilon^{jkl}(\partial_jN). \quad (130)$$

Equation (130) has split into three terms which we must now analyse.

The first term $\partial_j(N\epsilon^{ljk}e^a_lK^a_k)$ is a total derivative, which will vanish upon integration by parts and discarding of boundary terms. The second term of (130) contains the torsion of the connection Γ^a_i, which vanishes on account of (118). The third term of (130) is of a new character

$$K^a_ke^a_l\epsilon^{jkl}(\partial_jN) \rightarrow K_{kl}\epsilon^{jkl}(\partial_jN). \quad (131)$$

Equation (131) vanishes for $K_{kl} = K_{(kl)}$ symmetric in its indices, as is clear from (67) and (68). Hence we have upon transfering the triad to the opposite side and contracting with ∂_jN that

$$\beta\epsilon^{kij}K_{ji}(\partial_jN) = \beta E^k_a(\partial_kN)G_a \equiv \beta\theta^aG_a. \quad (132)$$

So we see, by virtue of the choice of definition of the connection A^a_i, that the terms linear in β vanish. Combining the results of (122), (124), (131), (130) and (132), we see that the smeared Hamiltonian constraint of the instanton representation (111) is given in metric variables by

$$\boldsymbol{H}_{Inst} = \int dt\int_\Sigma d^3x\sqrt{-g}\Big(\Lambda + \frac{1}{2}{}^{(3)}R[h] + \frac{\beta^2}{2}\big((\mathrm{tr}K)^2 - \mathrm{tr}K^2\big)\Big). \quad (133)$$

Equation (133) has a potential energy term ${}^{(3)}R$ and a kinetic energy term β^2VarK, where $VarK = (\mathrm{tr}K)^2 - \mathrm{tr}K^2$. Note the appearance of the Immirzi parameter in (133). The implication is that one must restrict to β either real or pure imaginary in order to have real metric gravity, which seems to rule out complex β.

5.4 Sign of the kinetic energy term

To complete the transition from the instanton into the metric representation, we must perform a Legendre transformation using (133) and the canonical structure to construct an action. First re-define π^{ij} as

$$\pi^{ij} = \beta\sqrt{h}\big(K^{ij} - h^{ij}\mathrm{tr}K\big), \quad (134)$$

which absorbs the parameter β into the definition of π^{ij} in lieu of (105). Then using (133), we can construct an action

$$I = \frac{1}{2}\int dt \int_{\Sigma} d^3x\Big(\pi^{ij}\dot{h}_{ij} - N\sqrt{h}\Big(2\Lambda + {}^{(3)}R[h] + \beta^2\big((\mathrm{tr}K)^2 - \mathrm{tr}K^2\big)\Big) \quad (135)$$

In what follows we will omit the factor of $\frac{1}{2}$, since it is common to all terms. From (135) one has

$$\frac{\delta I}{\delta \dot{h}_{ij}} = \pi^{ij}, \tag{136}$$

which identifies the 3-metric h_{ij} and the quantity π^{ij} as canonically conjugate variables. We must now write the Hamiltonian constraint in terms of π^{ij}. Inverting (134) we have

$$K_{ij} = \frac{1}{\beta\sqrt{h}}\big(\pi_{ij} - \frac{1}{2}h_{ij}(\mathrm{tr}\pi)\big); \quad \mathrm{tr}K = -\frac{1}{2\beta\sqrt{h}}(\mathrm{tr}\pi). \tag{137}$$

Then the following result ensues

$$\beta^2\big((\mathrm{tr}K)^2 - \mathrm{tr}K^2\big) = \frac{1}{h}\Big(\frac{1}{2}(\mathrm{tr}\pi)^2 - \pi_{ij}\pi^{ij}\Big), \tag{138}$$

such that the factor of β^2 cancels out from the Hamiltonian. The result is that the action can be written as

$$I = \frac{1}{2}\int dt \int_{\Sigma} d^3x\big(\pi^{ij}\dot{h}_{ij} - NH\big), \tag{139}$$

where the Hamiltonian constraint is given by

$$H = \frac{1}{2\sqrt{h}}\Big((\mathrm{tr}\pi)^2 - \pi_{ij}\pi^{ij}\Big) + \sqrt{h}{}^{(3)}R. \tag{140}$$

Since the factor of β^2 in (133) has cancelled out by being absorbed into the definition of the conjugate momentum, one sees that the reality of the metric theory is transparent to the possibility that β might be complex. But purely at the Lagrangian level, if we require A^a_i to be self-dual by choosing $\beta = i$ then this leads to two main implications for the metric representation: (i) The kinetic energy term of (140) acquires a minus sign, meaning that the π^{ij} becomes imaginary. This is a classically inaccessible configuration in metric GR, which would correspond to tunelling in the quantum theory. (ii) If one eliminates β from the definition (134), then one can also avoid the

appearance of a complex action by performing a Wick rotation of the lapse function $N \to iN$ and bring out the overall factor of i as in

$$I = \int dt \int_\Sigma d^3x i\pi^{ij}\dot{h}_{ij} - iNH = i \int dt \int_\Sigma d^3x \pi^{ij}\dot{h}_{ij} - NH. \qquad (141)$$

Then one can have a real action. Note, by generalization of the velocity of h_{ij} to its Lie derivative, one obtains the diffeomorphism constraint as previously shown, and one has the Hamiltonian form of the Einstein–Hilbert action.

6 Conclusion

The results of the present paper are as follows. We have shown that the configuration space of the instanton representation implies the existence of an intrinsic spatial geometry associated with nonabelian gauge theories in general. This intrinsic geometry, which for Yang–Mills theory corresponds to an Einstein space, has been generalized to incorporate gravitational degrees of freedom by way of the momentum space of the instanton representation. Upon having rewritten the gauge connection in terms of its spatial counterpart, we were able to establish this link through the corresponding curvatures. The decomposition of the gauge connection, in conjunction with the implementation of the Gauss' law constraint, makes clear the interpretation of the ensuing 3-geometry as a the manifestation of 4-dimensional one. Using the CDJ Ansatz, we have obtained the physical interpretation of the CDJ matrix in spatial terms as the Einstein tensor G_{ij} of this 3-dimensional space. We have also written the initial value constraints directly in terms of G_{ij}. Finally, we have also shown via the gauge theory route, how the Einstein–Hilbert action can be derived from the instanton representation. The result of this paper confirms the prescription from Paper II for constructing solutions to the Einstein equations. The 3-metric h_{ij} constructible from the instanton representation phase space is the same 3-metric which appears in the 3+1 ADM decomposition of the Einstein–Hilbert action. The result is that Einstein's general relativity can be written directly in terms of the physical degrees of the instanton representation. The advantage of this is that the physical degrees of freedom, which incorporate the initial value constraints (and particular the Gauss' law constraint in order to have a 4-dimensional metric theory of gravity), are more transparent in this latter representation whereas in terms of metric variables they are intractable. Hence the instanton representation of Plebanski gravity can be said to exhibit the physical degrees of freedom, while retaining the advantage afforded by the ploynomial form of the constraints in the Ashtekar formulation. We will utilize this feature in the quantization of the theory in separate works, beginning with Papers XV and XVIII.

7 Appendix A: 3+1 decomposition of spacetime

Let us start from the assumption that spacetime M has the topology of a globally hyperbolic Riemannian manifold with metric $g_{\mu\nu}$. Perform a 3+1 decomposition of M into $M = \Sigma \times R$, where Σ is a 3-dimensional spatial hypersurface labelled by a parameter t which marks the flow of time. This induces the following 3+1 decomposition of the spacetime metric $g_{\mu\nu}$,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -g_{00}dt^2 + 2g_{0i}dx^i dt + g_{ij}dx^i dx^j. \tag{142}$$

From (142) one reads off the metric coefficients, given in matrix form by

$$g_{\mu\nu} = \begin{pmatrix} N_i N^i - N^2 & N_j \\ N_i & g_{ij} \end{pmatrix}$$

and defines $N^\mu = (N, N^i)$, respectively the lapse function and the shift vector. The determinant of the spacetime metric $g_{\mu\nu}$ in terms of its 3+1 decomposition is given by

$$\sqrt{-g_{\mu\nu}} = N\sqrt{\det(g_{ij})}, \tag{143}$$

which defines a spacetime volume

$$Vol(M) = \int_\Sigma d^4x\sqrt{-g}. \tag{144}$$

One possible route in the transition to nonmetric gravity is to decompose $g_{\mu\nu}$ into tetrads e^I_μ, where I and μ respectively are Lorentz and spacetime indices, each of which take on the values $0, 1, 2, 3$. The spacetime metric in terms of tetrads is given by

$$g_{\mu\nu} = \eta_{IJ}e^I_\mu e^J_\nu, \tag{145}$$

where $\eta_{IJ} = Diag(-1, 1, 1, 1)$ is the Minkowski metric. At an unconstrained level e^I_μ contains 16 degrees of freedom per point and the metric $g_{\mu\nu}$ contains 10. The $16 - 10 = 6$ remaining degrees of freedom correspond to the freedom to perform $SO(3, 1)$ transformations which preserve η_{IJ}, and consequently the invariance of (145).

In what follows we will adopt the following 3+1 split $\mu = (0, i)$ and $I = (0, a)$, where i denotes spatial indices in 3-space Σ. The indices a denote internal indices in $SO(3, C)$, which is isomorphic to either $SU(2)$ subgroup. Let us also define spatial and inverse spatial triads e^a_i and E^i_a, such that

$$E_a^i e_j^a = \delta_j^i; \quad E_a^i E_a^j = H^{ij}. \tag{146}$$

In preserving the link from nonmetric to metric gravity we must verify that (145) is consistent with (142). Expanded into components with respect to η_{IJ}, we have for the time–time component

$$g_{00} = -(e_0^0)^2 + (e_0^a)^2 = -n^2 + (e_0^a)^2, \tag{147}$$

where we have defined $n = e_0^0$. The time–space component is given by

$$g_{0i} = -e_0^0 e_i^0 + e_0^a e_i^a = -n\beta_i + e_0^a e_i^a = N_i, \tag{148}$$

where N_i is the covariant form of the shift vector, and we have defined $\beta_i = e_i^0$. The spatial components are given by

$$g_{ij} = -e_i^0 e_j^0 + e_i^a e_j^a = -\beta_i \beta_j + e_i^a e_j^a. \tag{149}$$

It seems natural to associate the object $e_i^a e_j^a$ with the intrinsic 3-metric h_{ij} of a spatial slice Σ, then one could use $(h^{-1})^{ij} \equiv E_a^i E_a^j = H^{ij}$ to raise spatial indices, where $E_i^a = (e^{-1})_i^a$. However, we must regard g_{ij}, which is the actual spatial metric on Σ induced from $g_{\mu\nu}$, as being independent of β_i. So the dependence on β_i must be confined to e_i^a, and only for $\beta_i = 0$ can we identify h_{ij} with g_{ij}.

Let us eliminate e_0^a from (147) and (148), which is given by

$$e_0^a = E_a^i (n\beta_i + N_i). \tag{150}$$

Substitution of (150) into (147) yields

$$\begin{aligned} g_{00} &= -n^2 + E_a^j E_a^j (n\beta_i + N_i)(n\beta_j + N_j) \\ &= -n^2(1 - H^{ij}\beta_i\beta_j) + 2nH^{ij}\beta_i N_j + H^{ij} N_i N_j. \end{aligned} \tag{151}$$

Define $\vec{a} \cdot \vec{b} = H^{ij} a_i b_j$ as the inner product of two covariant 3-vectors with respect to the contravariant metric (146). Then (151) is given by

$$g_{00} = -n^2\big(1 - \vec{\beta} \cdot \vec{\beta}\big) + 2n\vec{\beta} \cdot \vec{N} + \vec{N} \cdot \vec{N} = -N^2 + \vec{N} \cdot \vec{N}. \tag{152}$$

In order for (152) to be consistent with the matrix form of the temporal component of (142) we must make identification

$$N^2 = n^2\big(1 - \vec{\beta}\cdot\vec{\beta}\big) - 2n\vec{\beta}\cdot\vec{N}, \tag{153}$$

which defines the lapse function N. Equation (153) is a quadratic equation for n with solution

$$n = \frac{\vec{\beta}\cdot\vec{N} \pm \sqrt{(\vec{\beta}\cdot\vec{N})^2 + N^2(1-\vec{\beta}\cdot\vec{\beta})}}{1-\vec{\beta}\cdot\vec{\beta}}. \tag{154}$$

Recall in the metric description that N parametrizes evolution of the dynamical variables in the direction normal to Σ and is independent of the shift vector $\vec{N}$, which parametrizes spatial diffeomorphisms within Σ. In the metric description of gravity N and $\vec{N}$ are independent functions, which constitutes four freely specifiable degrees of freedom corresponding to spacetime diffeomorphisms. Finally, note for $\vec{\beta}=0$, we have that $n=\pm N$.

7.1 Identification with Lorentz transformation parameters

Since the spacetime M of general relativity is a generalization of Minkowski spacetime $M^{(4)}$, it must be required as a matter of consistency that all curved spacetime quantities reduce to their flat spacetime counterparts in the appropriate limit. The group of spacetime diffeomorphisms is a main ingredient which distinguishes general from special relativity. It is clear from the 3+1 decomposition that this is related to the lapse-shift combination $N^\mu = (N, N^i)$, which constitute four independent functions on M. So there are two main consistency checks which the 3+1 decomposition must satisfy in the limit $M \rightarrow M^{(4)}$:
(i) The diffeomorphism group must become trivial, namely that $N^\mu \rightarrow (1,0)$. For equation (154) this implies that

$$\lim_{(N,\vec{N})\longrightarrow(1,0)} n = \pm\big(1-\vec{\beta}\cdot\vec{\beta}\big)^{-1/2}. \tag{155}$$

If we make the identification $e^0_i = \beta_i \equiv \frac{v_i}{c}$, corresponding to a Lorentz observer travelling with 3-velocity v_i, then $\big(1-\vec{\beta}\cdot\vec{\beta}\big)^{-1/2} \equiv \gamma$ takes on the physical interpretation of the Lorentz contraction factor where $\gamma > 1$.[9] Then $\vec{\beta}$ parametrizes the boost degrees of freedom contained in $SO(3,1)$.

This interpretation is borne out in (150), where one sees that β_i are independent functions separate from N_i. Consequently in the limit $M \rightarrow M^{(4)}$,

[9] This is a constraint from special relativity that places an upper bound of c on the speed of all Lorentz observers.

$$\lim_{(N,\vec{N})\longrightarrow(1,0)} e^a_0 = \gamma(E^i_a\beta_i) \equiv \gamma\beta_a. \tag{156}$$

(ii) It follows that in conjunction with (156), we must require the tetrad e^I_μ to reduce to a Lorentz transformation as $M \rightarrow M^{(4)}$, which can be seen as follows. Choose $g_{\mu\nu}$ in (145) to be the Minkowski metric $\eta_{\mu\nu}$, such that

$$\eta_{\mu\nu} = \eta_{IJ} e^I_\mu e^J_\nu. \tag{157}$$

Then we are expressing the flat Minkowsi metric in a frame where it is locally flat, which is also the case globally. This suggests the physical interpretation of the tetrad as a Lorentz transformation matrix for curved spacetime since in the limit $g_{\mu\nu} \rightarrow \eta_{\mu\nu}$. But recall that the Lorentz group in four dimensions is a six dimensional manifold embedded in a sixteen dimensional space, parametrized by three rotation and three boost parameters $(\vec{\theta}, \vec{\beta})$. The tetrad contains sixteen components, which naively suggests a sixteen dimensional manifold. But we will see that it is really 13 dimensional.
(iii) We have analyzed the implications for e^I_0 and e^0_μ in the limit $N \rightarrow M^{(4)}$, which leaves remaining the spatial components of the tetrad. It follows that in this same limit, the triads e^a_i must reduce to a $SO(3)$ rotation. So the next task is to account for the D.O.F. by examining in what manner they correlate to GR, which brings us to focus on the spatial part of (145)

$$e^a_i e^b_j \delta_{ab} = h_{ij} = g_{ij} + \beta_i\beta_j. \tag{158}$$

In the limit $M \rightarrow M^{(4)}$ we must have that $g_{ij} \rightarrow \delta_{ij}$, and that the part of (158) not involving $\vec{\beta}$ must reduce an orthogonal transformation of the unit three by three matrix δ_{ab}. Clearly the most general transformation under which δ_{ab} is invariant is a spatial rotation, since a boost is a non-unitary transformation. This implies that for $\vec{\beta} = 0$ then $e^a_i \rightarrow e^a_i(\vec{\theta})$ must reduce to the corresponding $SO(3)$ rotation matrix, which in Euclidean space contains three independent parameters $\vec{\theta}$. So we must ascertain to what extent this can possibly become generalized when one transitions to the non-Euclidean spaces of general relativity.

Since the induced 3-metric g_{ij} is symmetric in i and j, then when there exist three linearly independent eigenvectors it admits a polar decomposition

$$g_{ij} = (e^{\vec{\theta}\cdot\vec{J}})_{ia} \begin{pmatrix} g_1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{pmatrix}_{ab} (e^{-\vec{\theta}\cdot\vec{J}})_{bj}$$

which is parametrized by six parameters. Three parameters correspond to a rescaling of the axes of an orthogonal coordinate system by $(\sqrt{g_1}, \sqrt{g_2}, \sqrt{g_3})$

in the three respective directions. The remaining three parameters $\vec{\theta}$ correspond to a rotation of these axes. Since g_{ij} is real-valued and is symmetric in i and j, then the latter is an orthogonal transformation. In the limit $(\sqrt{g_1}, \sqrt{g_2}, \sqrt{g_3}) \to (1,1,1)$ and $\vec{\beta} \to 0$ we have

$$\delta_{ij} = (e^{\vec{\theta}\cdot\vec{J}})_{ia} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{ab} (e^{-\vec{\theta}\cdot\vec{J}})_{bj},$$

whence the internal and spatial indices $i \equiv a$ become indistinguishable. Therefore the scale factors $(\sqrt{g_1}, \sqrt{g_2}, \sqrt{g_3})$ break the rotational invariance which δ_{ab} possessed in Euclidean space. Equation (158) can be written as a polar decomposition

$$h_{ij} = (e^{\vec{\theta}\cdot J})_{in}\sqrt{g_n}(e^{-\vec{\theta}\cdot J})_{nj} + \beta_i\beta_j, \tag{159}$$

which is consistent with the requirement that the triad $e^a_i \equiv e^a_i(\vec{g},\vec{\theta},\vec{\beta})$ should contain nine degrees of freedom, three more than $g_{ij} = g_{ij}(\vec{g},\vec{\theta})$.

In the intrinsic frame, defined as the frame where $\vec{\theta} = \vec{\beta} = 0$, then the triad amounts to the rescaling of the axes of an orthonormal coordinate system by scale factors $(\sqrt{g_1}, \sqrt{g_2}, \sqrt{g_3})$. The final result is that the tetrad e^I_μ must meet the following requirements: (i) It must contain the six parameters of a Lorentz transformation $(\vec{\theta},\vec{\beta})$ in order to reduce to $\Lambda^I_J(\vec{\theta},\vec{\beta})$ in the $M^{(4)}$ limit of M. (ii) It must contain the four parameters of the diffeomorphism group $N^\mu = (N, N^i)$ in order for its square to produce a spacetime metric $g_{\mu\nu}$. (iii) The spatial part of e^I_μ must contain six degrees of freedom corresponding to the 3-metric g_{ij}. Morover, these D.O.F. must be encoded in a Lorentz spatial rotation by $\vec{\theta}$ and a rescaling which is independent of this rotation. Hence we must have

$$e^I_\mu = \begin{pmatrix} n(\vec{\beta}, N^\mu) & \vec{\beta} \\ e^{-\vec{\theta}\cdot J}(n\vec{\beta} + \vec{N}) & e^a_i(\vec{g},\vec{\theta},\vec{\beta}) \end{pmatrix}.$$

Let us now compare a Lorentz transformation Λ^I_J with the tetrad e^I_μ by correlating degrees of freedom. A Lorentz transformation contains six D.O.F.. The three boost parameters correlate to $\beta_i \equiv e^0_i$. The rotation parameters correspond to the three rotational D.O.F. in the spatial triad e^a_i. This exhausts the 6 D.O.F. of the $SO(3,1)$ transformations on Minkowski spacetime. The remaining parts of the tetrad inlude three D.O.F. in the triad not including rotations, and the components e^0_0 and e^a_0. The freedom to rescale the axes of e^a_i by $\sqrt{g_a}$ exhausts the first three D.O.F., which we will regard as physical. The components e^a_0 from (150) correlate to the shift vector N_i, which is a nonphysical degree of freedom.

From (153), the signature of spacetime is determined by the projection of the boost vector $\vec{\beta}$ on to the shift vector N^i, since

$$
\begin{aligned}
n &> \frac{2\vec{\beta}\cdot\vec{N}}{1-\vec{\beta}\cdot\vec{\beta}} \text{ for Lorentzian signature spacetime;} \\
n &< \frac{2\vec{\beta}\cdot\vec{N}}{1-\vec{\beta}\cdot\vec{\beta}} \text{ for Euclidean signature spacetime;} \\
n &= \frac{2\vec{\beta}\cdot\vec{N}}{1-\vec{\beta}\cdot\vec{\beta}} \text{ for transition into signature change.}
\end{aligned} \tag{160}
$$

8 Appendix B: Self-dual two form decomposition

According to [?], [?], the information contained in the spacetime metric can equally well be encoded within a triple of self-dual two forms. For notational purposes we define self-dual $SO(3,C)$ valued two forms $\Sigma^a = \frac{1}{2}\Sigma^a_{\mu\nu}dx^\mu \wedge dx^\nu$, constructed from tetrads e^I_μ in the following manner

$$\Sigma^a = \Sigma^a_{0i}dt \wedge dx^i + \Sigma^a_{ij}dx^i \wedge dx^j = ie^0 \wedge e^a - \frac{1}{2}\epsilon^{abc}e^b \wedge e^c. \tag{161}$$

The two forms Σ^a arrange the same tetrads e^I_μ used in (145) into a self-dual combination, which discards one $SU(2)$ half of $SO(3,1)$. The self-dual two forms (161) satisfy the relation

$$\frac{1}{2}\Sigma^f \wedge \Sigma^g = \delta^{fg}\sqrt{-g}d^4x, \tag{162}$$

which fixes the volume element of spacetime. We will now expand (161) in terms of its constituent tetrad one forms, given by[10]

$$\begin{aligned} e^0 &= e^0_\mu dx^\mu = e^0_0 dt + e^0_i dx^i = ndt + \beta_i dx^i; \\ e^a &= e^a_\mu dx^\mu = e^a_0 dt + e^a_i dx^i = E^{aj}(N_j + \beta_j)dt + e^a_i dx^i. \end{aligned} \tag{163}$$

There are two contributions to (161). The first contribution is given by

$$\begin{aligned} e^0 \wedge e^a &= \big(ndt + \beta_i dx^i\big) \wedge \big(e^a_0 dt + e^a_j dx^j\big) \\ &= \big(ne^a_j - \beta_j e^a_0\big)dt \wedge dx^j + \beta_i e^a_j dx^i \wedge dx^j, \end{aligned} \tag{164}$$

and the second contribution is given by

$$\begin{aligned} \frac{1}{2}\epsilon^{abc}e^b \wedge e^c &= \frac{1}{2}\epsilon^{abc}\big(e^b_0 dt + e^b_j dx^j\big) \wedge \big(e^c_0 dt + e^c_k dx^k\big) \\ &= \epsilon^{abc}e^b_0 e^c_j dt \wedge dx^j + \frac{1}{2}\epsilon^{abc}e^b_j e^c_k dx^j \wedge dx^k. \end{aligned} \tag{165}$$

Combining the results of (164) and (165), we obtain

$$\Sigma^a = \Big(ine^a_j - e^b_0\big(i\delta_{ab}\beta_j + \epsilon^{abc}e^c_j\big)\Big)dt \wedge dx^j + \big(i\beta_j e^a_k - \frac{1}{2}\epsilon^{abc}e^b_j e^c_k\big)dx^j \wedge dx^k. \tag{166}$$

Substituting (150) into (166), this yields

[10] We have left in β_i to maintain generality. We will eventually set this to zero in congruity with the elimination of half of the Lorentz group.

$$\Sigma^a = \Big(ine^a_j - E^i_b(n\beta_i + N_i)\big(i\delta_{ab}\beta_j + \epsilon^{abc}e^c_j\big)\Big)dt \wedge dx^j$$
$$+\big(i\beta_j e^a_k - \frac{1}{2}\epsilon^{abc}e^b_j e^c_k\big)dx^j \wedge dx^k$$
$$= \Big(n\big(ie^a_j - iE^i_a\beta_i\beta_j - \epsilon^{abc}E^i_b\beta_i e^c_j\big) - N_i E^i_b\big(i\delta_{ab}\beta_j + \epsilon^{abc}e^c_j\big)\Big)dt \wedge dx^j$$
$$+\big(i\beta_j e^a_k - \frac{1}{2}\epsilon^{abc}e^b_j e^c_k\big)dx^j \wedge dx^k. \quad (167)$$

It is convenient to make a few definitions, which brings in the Ashtekar momentum space variables ([4], [5], [6]). Define a densitized triad $\widetilde{\sigma}^i_a$ by

$$\widetilde{\sigma}^i_a \equiv \frac{1}{2}\epsilon^{ijk}\epsilon_{abc}e^b_j e^c_k = (\text{det}e)(e^{-1})^i_a. \quad (168)$$

Inversion of (168) yields the relation[11]

$$e^a_i = (\widetilde{\sigma}^{-1})^a_i(\text{det}\widetilde{\sigma})^{1/2} = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}^j_b\widetilde{\sigma}^k_c(\text{det}\widetilde{\sigma})^{-1/2}, \quad (169)$$

as well as

$$E^i_a = (\text{det}\widetilde{\sigma})^{-1/2}\widetilde{\sigma}^i_a = (e^{-1})^i_a \quad (170)$$

and

$$(\widetilde{\sigma}^{-1})^a_i = (\text{det}E)(E^{-1})^a_i = \frac{1}{2}\epsilon^{abc}\epsilon_{ijk}E^j_b E^k_c. \quad (171)$$

Additionally, the following relation holds

$$\sqrt{h} = \sqrt{\text{det}\widetilde{\sigma}} = (\text{det}e) = (\text{det}E)^{-1}. \quad (172)$$

Let us now construct a 3-metric using the triads $h_{ij} = e^a_i e^a_j$. In terms of the Ashtekar variables this is given by

$$h_{ij} \equiv e^a_i e^a_j = (\widetilde{\sigma}^{-1})^a_i(\widetilde{\sigma}^{-1})^a_j(\text{det}\widetilde{\sigma}). \quad (173)$$

Inverting (173) we obtain

[11] We have restricted ourselves to a nondegenerate triad, which excludes topology changing configurations from our consideration.

$$\widetilde{\sigma}_a^i \widetilde{\sigma}_a^j (\det\widetilde{\sigma})^{-1} = E_a^i E_a^j = h^{ij} \longrightarrow \widetilde{\sigma}_a^i \widetilde{\sigma}_a^j = h h^{ij}. \tag{174}$$

Recall that g_{ij} is the actual spatial metric induced from the spacetime metric $g_{\mu\nu}$, and the 3-metric h_{ij}, which in general may contain boost degrees of freedom, is given by

$$h_{ij} = g_{ij} + \beta_i \beta_j. \tag{175}$$

The determinant of g_{ij} can be expressed in terms of the determinant of h_{ij} using the expansion

$$\det(g_{ij}) = \frac{1}{6}\epsilon^{ijk}\epsilon^{lmn}\Big(h_{il}h_{jm}h_{kn} - 3h_{il}h_{jm}\beta_k\beta_n + 3h_{il}\beta_j\beta_m\beta_k\beta_n - \beta_i\beta_l\beta_j\beta_m\beta_k\beta_n\Big). \tag{176}$$

The third and fourth terms of (176) vanish due to antisymmetry, and we are left with

$$\det(g_{ij}) = (\det h_{ij})\big(1 - H^{kn}\beta_k\beta_n\big) \longrightarrow \sqrt{{}^{(3)}g} = \sqrt{1 - \vec{\beta}\cdot\vec{\beta}}\sqrt{h}. \tag{177}$$

Hence the induced 3-metric and corresponding spatial volume element for g_{ij} coincides with that of h_{ij} in a particular Lorentz frame where the boost degrees of freedom β_i are set to zero. The restriction $\beta_i = 0$ is known as the time gauge.

Let us now write the two form Σ^a in terms of the Ashtekar variables. This is given by

$$\Sigma^a = \bigg(N\gamma\Big(\frac{1}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k(\det\widetilde{\sigma})^{-1/2} + \widetilde{\sigma}_b^j\Big(-\delta_{ab}\beta_i\beta_j(\det\widetilde{\sigma})^{-1/2} + \epsilon_{abc}\beta_i(\widetilde{\sigma}^{-1})_j^c\Big) + N_i\big(-\delta_{ab}\beta_j + \epsilon_{abc}(\widetilde{\sigma}^{-1})_j^c(\det\widetilde{\sigma})^{1/2}\Big)dt\wedge dx^j + \Big(\beta_i(\widetilde{\sigma}^{-1})_j^a\det\widetilde{\sigma})^{1/2} + \epsilon_{ijk}\widetilde{\sigma}_a^i\Big)dx^j\wedge dx^k \tag{178}$$

Equation (178) is the full two form, which includes the boost parameters $\vec{\beta}$. Certainly $\vec{\beta} = 0$ is allowed, a specific case where triadic to metric equivalence exists. In the limit $\beta_i \to 0$ then (178) reduces to

$$\Sigma^a = \bigg(N\Big(\frac{i}{2}\epsilon_{ijk}\epsilon^{abc}\widetilde{\sigma}_b^j\widetilde{\sigma}_c^k(\det\widetilde{\sigma})^{-1/2} - N_i\big(\epsilon_{abc}(\widetilde{\sigma}^{-1})_j^c(\det\widetilde{\sigma})^{1/2}\Big)dt\wedge dx^j - \epsilon_{ijk}\widetilde{\sigma}_a^i dx^j\wedge dx^k, \tag{179}$$

which is the starting point for the Ashtekar description in the time gauge. The volume form of spacetime is given by

$$\begin{aligned}&\big(i\beta_i e^a_j - \frac{1}{2}\epsilon^{abc} e^b_i e^c_j\big)\Big(n\big(ie^a_k - iE^m_a\beta_m\beta_k - \epsilon^{abd}E^m_b\beta_m e^d_k\big)\\&\quad -N_m E^m_b\big(i\delta_{ab}\beta_k + \epsilon^{abd}e^d_k\big)\Big) dt \wedge dx^i \wedge dx^j \wedge dx^k.\end{aligned} \tag{180}$$

Equation (180) should be consistent with the volume form in metric variables

$$\sqrt{-{}^{(4)}g}d^4x = N\sqrt{-{}^{(3)}g}d^4x, \tag{181}$$

which enables us to solve for n.

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